**Total Unimodularity**

A square integer matrix is unimodular if its determinant is 1 or -1

An integer matrix is totally unimodular if all of its non-singular submatrices is unimodular

There are two sufficient conditions for a matrix consisting of only (0, +1, -1) to be totally unimodular:

**Condition 1:** Each column contains at most two non-zero elements

**Condition 2:** The rows of A can be partitioned into two set: A1 and A1 such that the two nonzero entries are in the same set if they have different signs and in different sets if they have the same sign.

The following is an example of a matrix that satisfies conditions 1 and 2

|  |  |  |
| --- | --- | --- |
| 1 | 0 | 0 |
| 1 | 1 | -1 |
| 0 | -1 | 1 |

Clearly this satisfies condition 1; can we make it satisfy condition 2? Yes, let us swap row 2 and 3 and we get:

|  |  |  |
| --- | --- | --- |
| 1 | 0 | 0 |
| 0 | -1 | 1 |
| 1 | 1 | -1 |

Where the red cells are set A1 and the blue cells are set A2

So let’s take it one step further and make the statement that any (0,+1,-1) rank n matrix that has exactly one +1 in each column and one -1 in each column with all other entries being zero is totally unimodular. Why is that? Because it is ALWAYS possible to separate A into two sets A1 and A2:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| -1 | -1 | 0 | 0 | 0 | 1 |
| 1 | 0 | -1 | -1 | 0 | 0 |
| 0 | 1 | 1 | 0 | -1 | 0 |
| 0 | 0 | 0 | 1 | 1 | -1 |

Where the red cells are A1 and set A2 is the empty set.

So why is this important? Lets look at the definition of an inverse matrix

A-1 = 1/det(A) \* Adjoint of A

Where the Adjoint of A is the cofactor of A transposed

Where the cofactor of A is a matrix of determinant values for all 2 X 2minors of A

So if all determinants in a totally unimodular matrix = 1, -1, then all terms in the cofactor of A will be 1, -1 (integers)

If all terms in the cofactor matrix of A are integers, then the terms in the adjoint of A will be an integer

If the terms of the adjoint are integer, and the determinant of a non-singular square sub matrix of A is (1, -1), then all of the terms in A-1 are integer

Which is nice, but why is that important? Lets look at the standard LP

MIN cTx

ST Ax=b

x > 0

We know any feasible solution to this problem can be broken into the basis matrix and non-basis matrix (where the basis matrix contains the basic variables – those variables active in the solution, and the non-basis matrix containing those variables not in the solution). In other words:

Ax = BxB + NxN

Which means the constraints can be written as

BxB + NxN = b

We also know the non-basic terms are all zero so the solution goes to:

BxB = b

Solving for xB yields

xB = B-1b

So if A is totally unimodular, we know B is unimodular, which means its inverse is integral, which means if the vector b is integral, then the solution will be integral.

So the bottom line is that any LP that has a totally unimodular constraint matrix that has integral RHS will have an integer solution (all vertices of the polyhedron that define the feasible region reside on integer points)

So what does this mean to us in pure network optimization problems?

If you look at the LP formulation of network flow problems, the constraint matrix is really the node-edge incidence matrix. This matrix has n rows and m columns, where each row represents a node and each column represents an edge. Each edge has a tail node (represented by a +1) and a head node (represented by a -1). Since an edge can only connect two nodes, all other entries are 0. So we have a rank(n) (0, 1, -1) integer matrix with exactly one (+1) and one (-1) and all others are (0). This is exactly the structure specified earlier that says we are guaranteed to have a totally unimodular matrix. That means all solutions to a network flow problem will be integer if the right hand side values are integer.